Extended Abstract: Reconsidering Intuitionistic Duality

Aaron Stump, Harley Eades III, Ryan McCleeary
Computer Science
The University of Iowa

1 Introduction

This paper proposes a new syntax and proof system called Dualized Intuitionistic Logic (DIL), for intuitionistic propositional logic with the subtraction operator. Our goal is a conservative extension of standard propositional intuitionistic logic with perfect duality (symmetry) between positive and negative connectives. The proof system should satisfy the following metatheoretic properties: soundness, completeness, cut elimination, and substitution. To our knowledge, no existing system achieves these goals. Substitution is needed for cut elimination, but has posed problems for other systems; for example Crolard develops a complex dependency-tracking calculus to obtain substitution for a constructive type theory with subtraction [3].

In this extended abstract, we describe our work in progress on DIL. We have formulated a dualized syntax and proof system, for which we have proved soundness with respect to a standard Kripke semantics for intuitionistic propositional logic. We have also proved substitution. Regarding completeness, we point out an issue in prior work: we exhibit a semantically valid formula which is provable in DIL but which lacks, under the obvious translation, a cut-free proof in the system SLK[1] of Crolard [2]. This shows that Crolard's system cannot satisfy both cut elimination and completeness.

Our motivation is to obtain a new logical foundation for type theory, with a perfect duality between positive and negative computation. Computational classical type theories (CCTTs) like those proposed in [9] and [4] exhibit such a duality, but are not suitable for computation since they, like other CCTTs, lack the canonicity property: closed terms of type T are not necessarily built with a constructor for T (e.g., pairing for $T \wedge T'$). For other work seeking to support control and canonicity, see [5] and works cited there. Like some CCTTs, DIL is formulated as a sequent calculus. In future work, we intend to use the Curry-Howard isomorphism to develop a Dualized Type Theory based on DIL, where positive assumptions will become input variables and negative ones output variables. Cut will become a control operator much like the μ-operator of Curien and Herbelin's system [4]. We thus propose to explore intuitionistic duality as a basis for constructive control, in contrast to (non-constructive) CCTTs.

But the main benefit we are seeking from a completely dualized system is a uniform simultaneous treatment of induction and coinduction. It is well known that induction and coinduction are duals semantically. While this duality has been considered in CCTT [7], no constructive system with induction and coinduction as duals has yet been proposed. Proof assistants like Coq and Agda have unsatisfactory treatments of (mixed) induction and coinduction (see the discussion in [1]): Coq lacks type preservation in the presence of coinductive types, a serious defect in the system, while Agda restricts how inductive and coinductive types can be nested. Our working hypothesis is that a logical foundation based on intuitionistic duality will allow the semantic duality between induction and coinduction to be expressed in type theory, yielding a solution to the problems with these important features in existing systems. Detailed proofs of all lemmas and theorems below may be found in a companion document on the first author's web site.

2 Syntax of Dualized Intuitionistic Logic (DIL)

$$\begin{array}{lcl} \textit{polarities } p & ::= & + \mid - \\ \textit{formulas } T & ::= & A \mid \langle p \rangle \mid T \to_p T' \mid T \wedge_p T' \end{array}$$

Figure 1: Syntax of formulas for DIL

The syntax for polarities p and then formulas T of Dualized Intuitionistic Logic (DIL) is given in Figure 1. We write $\bar{p}$ for the opposite polarity from p (so $\bar{+} = -$ and $\bar{-} = +$). The formulas A are drawn from a set $\mathcal{A}$ of atomic propositional formulas. As the semantics below will make precise, the logical constructs above can be identified with standard ones as follows: $\langle + \rangle$ is *True*, $\langle - \rangle$ is *False*, $T \wedge_+ T'$ is $T \wedge T'$, $T \wedge_- T'$ is $T \vee T'$, $T \to_+ T'$ is $T \to T'$, and $T \to_- T'$ is subtraction $T' - T'$ (note the reversed order of subformulas).

3 Kripke Semantics for DIL

We will work with standard Kripke models $(W, \preceq, V)$ (cf. Chapter 7 of [8]), where W is a non-empty set of objects called worlds, $\preceq$ is a preorder on W called the accessibility relation, and V maps each world w in W to a subset of $\mathcal{A}$, namely the atomic formulas which are true in w. As standard, V is required to be monotonic: for all $w \in W$ and $A \in \mathcal{A}$, if $A \in V(w)$, then $A \in V(w')$ for all $w' \succeq w$. Figure 2 defines a semantics $[\![T]\!]_w$ relative to a Kripke model $(W, \preceq, V)$, to interpret a formula T in a world $w \in W$. The semantics exactly follows standard semantics for intuitionistic propositional logic with subtraction (cf. [6]).

Theorem 1 (Monotonicity)**.** *Suppose $w \preceq^p w'$. Then $p[\![T]\!]_w$ implies $p[\![T]\!]_{w'}$.*

Relational notation. We write $\preceq^p$ to indicate $\preceq$ if $p = +$, and $\succeq$ if $p = -$.

Paths. We define a bi-directional path π of a Kripke model to be a possibly empty list of worlds $w_1, \cdots, w_n$ such that for all $i \in \{1, \ldots, n-1\}$, we have either $w_i \preceq w_{i+1}$ or $w_i \succeq w_{i+1}$. For such a path, we denote its length (n) by $|\pi|$. We will write $\pi \preceq^p w$ to mean that $\pi = \pi', w'$ for some π' and w', with $w' \preceq^p w$. We do not distinguish a singleton path w from the world w.

4 A Proof System for DIL

In this section we define a proof system for DIL, which we will subsequently justify using the modal semantics defined above. The derivable objects of the proof system are sequents of the form $\Gamma \vdash^p T$, where Γ is a modal context as defined in Figure 3.

$$\begin{array}{lcl} [\![A]\!]_w & \Leftrightarrow & A \in V(w) \\ [\![\langle + \rangle]\!]_w & \Leftrightarrow & \textit{true} \\ [\![\langle - \rangle]\!]_w & \Leftrightarrow & \textit{false} \\ [\![T \to_+ T']\!]_w & \Leftrightarrow & \forall w'.w \preceq w' \Rightarrow [\![T]\!]_{w'} \Rightarrow [\![T']\!]_{w'} \\ [\![T \to_- T']\!]_w & \Leftrightarrow & \exists w'.w \succeq w' \wedge \neg[\![T]\!]_{w'} \wedge [\![T']\!]_{w'} \\ [\![T \wedge_+ T']\!]_w & \Leftrightarrow & [\![T]\!]_w \wedge [\![T']\!]_w \\ [\![T \wedge_- T']\!]_w & \Leftrightarrow & [\![T]\!]_w \vee [\![T']\!]_w \end{array}$$

Figure 2: Semantics of DIL formulas

$$\begin{array}{lcl} \textit{local contexts}\ \Delta & ::= & \cdot \mid pT, \Delta \\ \textit{modal contexts}\ \Gamma & ::= & \Delta \mid \Gamma \preceq^p \Delta \end{array}$$

Figure 3: Logical contexts for DIL

$$\frac{}{\Gamma \preceq^{p'} \Delta_1, pT, \Delta_2 \vdash^p T}\ \textsc{ax} \qquad \frac{\Gamma \vdash^p T}{\Gamma \preceq^p \Delta \vdash^p T}\ \textsc{weak} \qquad \frac{}{\Gamma \vdash^p \langle p \rangle}\ \textsc{unit}$$

$$\frac{\Gamma \preceq^p pT \vdash^p T'}{\Gamma \vdash^p T \to_p T'}\ \textsc{imp} \qquad \frac{\begin{array}{c}\Gamma \vdash^{\bar{p}} T \\ \Gamma \vdash^p T'\end{array}}{\Gamma \vdash^p T \to_{\bar{p}} T'}\ \textsc{impBar} \qquad \frac{\begin{array}{c}\Gamma \vdash^p T_1 \\ \Gamma \vdash^p T_2\end{array}}{\Gamma \vdash^p T_1 \wedge_p T_2}\ \textsc{and}$$

$$\frac{\Gamma \vdash^p T_1}{\Gamma \vdash^p T_1 \wedge_{\bar{p}} T_2}\ \textsc{andBar1} \qquad \frac{\Gamma \vdash^p T_2}{\Gamma \vdash^p T_1 \wedge_{\bar{p}} T_2}\ \textsc{andBar2} \qquad \frac{\begin{array}{c}\Gamma, \bar{p}T \vdash^p T' \\ \Gamma, \bar{p}T \vdash^{\bar{p}} T'\end{array}}{\Gamma \vdash^p T}\ \textsc{cut}$$

Figure 4: Proof Rules for DIL

4.1 Contexts

Intuitively, a local context Δ describes a world, while a modal context describes a (bi-directional) path. The starting point of the path corresponds to the leftmost local context in the modal context. We treat pT as an abbreviation for local context $pT, \cdot$. We sometimes also view contexts as built from right to left instead of left to right. We concatenate local contexts with Δ, Δ', and modal contexts with $\Gamma \preceq^p \Gamma'$. We do not distinguish a modal context of the form Δ from the local context Δ. The local extension Γ, Δ of modal context Γ by local context Δ is defined as follows:

$$(\Gamma' \preceq^p \Delta'), \Delta = \Gamma \preceq^p (\Delta', \Delta)$$

The local concatenation Γ, Γ' of modal contexts Γ and Γ' is then defined by:

$$\Gamma, (\Delta \preceq^p \Gamma'') = (\Gamma, \Delta) \preceq^p \Gamma''$$

When $\Gamma = \Delta$ or $\Gamma = \Gamma' \preceq^p \Delta$, we call Δ the current local context of Γ. It describes the abstract world w at the end of the abstract path described by Γ. We sometimes refer informally to that w as the current local world of Γ. The length $|\Gamma|$ of modal context Γ is the number of maximal local contexts contained in Γ; so, 1 plus the number of occurrences of $\preceq^+$ or $\preceq^-$. The use of an ordering in the context may also be found in the display calculus δ**BiInt** of Goré [6].

4.2 Proof system

Let us write $p\ F$ as meta-notation meaning F if $p \equiv +$ and $\neg F$ if $p \equiv -$. The intuitive meaning of the sequents, which we will make precise in the next section, is: $\Gamma \vdash^+ T$ iff in the world w at the end of the path determined by Γ, $p[\![T]\!]_w$ holds. The crucial idea of the proof system is to incorporate Theorem 1 (Monotonicity), in the WEAK rule: if we are following edges forward in the accessibility relation, then true formulas will remain true; and dually, if we follow edges backwards, false formulas will remain false.

Figure 4 gives the proof rules for deriving sequents $\Gamma \vdash^p T$. The rules allow expansion of the context in two different ways. If we expand the context locally, as in the CUT rule, we are adding a new assumption about the current world. If we expand the context modally, as in the IMP rule, we are extending our path π to a new world w' where $\pi \preceq^p w'$, and the assumed formula holds.

$$
\begin{array}{lcl}
[\![p\,T, \Delta]\!]_w & = & p\ [\![T]\!]_w \ \wedge\ [\![\Delta]\!]_w \\
[\![\cdot]\!]_w & = & \mathit{true} \\
\\
[\![\Delta]\!]_{\pi,w} & = & [\![\Delta]\!]_w \\
[\![T]\!]_{\pi,w} & = & [\![T]\!]_w \\
\\
[\![\Gamma \ \preceq^p\ \Delta]\!]_{\pi,w} & = & [\![\Gamma]\!]_\pi \ \wedge\ \pi \preceq^p w \ \wedge\ [\![\Delta]\!]_w \\
\\
[\![\Gamma \ \vdash^p\ T]\!]_\pi & = & [\![\Gamma]\!]_\pi \ \Rightarrow\ p[\![T]\!]_\pi
\end{array}
$$

Figure 5: Semantics of Local Contexts and Sequents

5 Semantics and Metatheory for DIL

For purposes of this section, fix an arbitrary Kripke model $(W, \preceq, V)$. Figure 5 defines a semantics for sequents, after first defining several helper predicates: $[\![\Delta]\!]_w$ expresses that world w satisfies local context Δ; $[\![\Delta]\!]_\pi$ expresses that the last world in the path π satisfies Δ; and $[\![\Gamma]\!]_\pi$ expresses that path π satisfies modal context Γ. The interpretation $[\![\Gamma \vdash^p T]\!]_\pi$ of sequents with respect to a path π is then defined using those predicates: $\Gamma \vdash^p T$ holds along path π iff assuming π satisfies Γ, then T holds in the last world of π.

Theorem 2 (Soundness)**.** *If $\Gamma_a \vdash^p T_a$ is derivable using the rules of Figure 4, and if π is a path where $|\pi| = |\Gamma_a|$ (the lengths of the path and the context are the same), then $[\![\Gamma_a \vdash^p T_a]\!]_\pi$.*

Theorem 3 (Substitution)**.** *If $\Gamma_1, p_1\,T_1, \Gamma_2 \vdash^{p_2} T_2$ and $\Gamma_1 \vdash^{p_1} T_1$, then $\Gamma_1, \Gamma_2 \vdash^{p_2} T_2$.*

6 Possible Incompleteness of Other Logics

Consider the following formula

$$A \to_+ (A \to_- A \to_+ \langle - \rangle) \to_- \langle + \rangle \qquad (\star_1).$$

Using the usual translation to classical logic one will see that this formula is an embedding of the law of excluded middle into intuitionistic logic. It is valid with respect to the semantics given in Section 3.

Lemma 4. *Suppose $M = \langle W, \preceq, V \rangle$ is a Kripke model, then for all worlds $w \in W$ we have $[\![A \to_+ (A \to_- A \to_+ \langle - \rangle) \to_- \langle + \rangle]\!]_w$.*

Proof. By definition we must show that $\forall w_1.(w \preceq w_1 \text{ and } [\![A]\!]_{w_1}) \Rightarrow [\![(A \to_- (A \to_+ \langle - \rangle)) \to_- \langle + \rangle]\!]_{w_1}$. Now suppose $w_1 \in W$ such that $w \preceq w_1$ and $[\![A]\!]_{w_1}$. Then we must show $\exists w_2.w_2 \preceq w_1$ and $\neg[\![A \to_- (A \to_+ \langle - \rangle)]\!]_{w_2}$ and $[\![\langle + \rangle]\!]_{w_2}$. Take w_1 for w_2. Clearly, $[\![\langle + \rangle]\!]_{w_1}$ holds. To show $\neg[\![A \to_- (A \to_+ \langle - \rangle)]\!]_{w_1}$ we must show $\forall w_3.w_3 \preceq w_1$ and $\neg[\![A]\!]_{w_3} \Rightarrow \neg[\![A \to_+ \langle - \rangle]\!]_{w_3}$. So assume $w_3 \in W$, $w_3 \preceq w_1$, and $\neg[\![A]\!]_{w_3}$. Then we must show $\neg[\![A \to_+ \langle - \rangle]\!]_{w_3}$. It suffices to show $\exists w_4.w_3 \preceq w_4$ and $[\![A]\!]_{w_4}$ and $\neg[\![\langle - \rangle]\!]_{w_4}$. Take w_1 for w_4. Clearly, $\neg[\![\langle - \rangle]\!]_{w_1}$, and by assumption we have $[\![A]\!]_{w_1}$. Therefore, $[\![A \to_+ (A \to_- A \to_+ \langle - \rangle) \to_- \langle + \rangle]\!]_w$. □

In addition, this formula has an easy cut-free derivation (elided) in DIL.

In [2], Crolard defines an intuitionistic logic with subtraction called SLK[1] which he states is complete with respect to the standard Kripke semantics for intuitionistic logic. This system uses the following forms of implication-left and subtraction-right rules, where the opposite context is required to be empty:

$$\frac{\Gamma, B \vdash A}{\Gamma \vdash B \Rightarrow A} \qquad \frac{A \vdash \Delta, B}{A - B \vdash \Delta}$$

We show next that the formula ($\star_1$) has no cut-free proof in SLK^1. We first must translate ($\star_1$) into the language of SLK^1. Its equivalent form is the following:

$$A \to (True - ((A \to False) - A)) \quad (\star_2).$$

Then using Crolard's definitions ($\star_2$) has a shorter form $A \to \sim(\neg A - A)$.

Theorem 5. *The formula $A \to (True - ((A \to False) - A))$ has no cut-free proof in* SLK^1*.*

The previous theorem shows that SLK^1 cannot satisfy both completeness (which follows directly from Theorem 2.4.3 and Proposition 4.4.1 of [2]) and cut elimination. It is unclear if the previous formula is derivable in SLK^1 if cut is used. Note that Goré's δ**BiInt** [6] does not have any restriction corresponding to that in the implication-right and subtraction-left rules of SLK^1, so we conjecture this formula is derivable in δ**BiInt** (due to lack of experience with display calculi, we have not been able to confirm this yet).

7 Conclusion

We have proposed Dualized Intuitionistic Logic (DIL), which is sound with respect to a standard Kripke semantics for propositional intuitionistic logic with subtraction, and has the substitution property. The crucial idea is to use modal contexts Γ to describe bi-directional paths, and incorporate monotonicity in the proof system. We plan to complete the metatheoretic analysis of DIL, in particular completeness and cut elimination. *Axiom cuts*, where one premise is a weakening (iterated application of WEAK) of AX cannot be eliminated, but we conjecture that all other cuts can be. We then plan to develop DIL into a Dualized Type Theory.

References

[1] Andreas Abel, Brigitte Pientka, David Thibodeau, and Anton Setzer. Copatterns: programming infinite structures by observations. In *Proceedings of the 40th annual ACM SIGPLAN-SIGACT symposium on Principles of Programming Languages (POPL)*, pages 27–38. ACM, 2013.

[2] Tristan Crolard. Subtractive logic. *Theor. Comput. Sci.*, 254(1-2):151–185, 2001.

[3] Tristan Crolard. A Formulae-as-Types Interpretation of Subtractive Logic. *J. Log. Comput.*, 14(4):529–570, 2004.

[4] P. Curien and H. Herbelin. The duality of computation. In *Proceedings of the fifth ACM SIGPLAN International Conference on Functional Programming (ICFP)*, pages 233–243. ACM, 2000.

[5] H. Geuvers, R. Krebbers, and J. McKinna. The lambda-mu-t-calculus. *Annals of Pure and Applied Logic*, 164(6):676–701, 2012.

[6] Rajeev Gor. Dual Intuitionistic Logic Revisited. In Roy Dyckhoff, editor, *Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX)*, pages 252–267. Springer, 2000.

[7] Daisuke Kimura and Makoto Tatsuta. Dual Calculus with Inductive and Coinductive Types. In Ralf Treinen, editor, *Rewriting Techniques and Applications (RTA)*, pages 224–238, 2009.

[8] G. Mints. *A Short Introduction to Intuitionistic Logic*. Kluwer Academic / Plenum Publishes, 2000.

[9] P. Wadler. Call-by-Value Is Dual to Call-by-Name – Reloaded. In Jürgen Giesl, editor, *Rewriting Techniques and Applications (RTA)*, Lecture Notes in Computer Science, pages 185–203. Springer, 2005.