On the categorical structure of bi-intuitionistic logics

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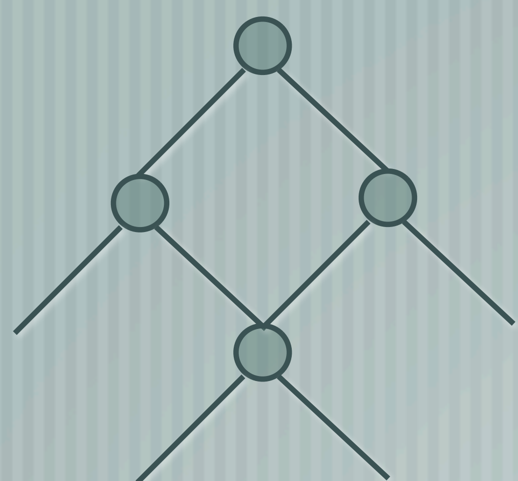
The Big-Big Picture

Inductive Data:



data Tree (A : Type) : Type where
  Leaf : A → Tree A
  Node : Tree A → Tree A → Tree A

Coinductive Data:

codata Stream (A : Type) : Type where
  cons : A → Stream A → Stream A

The Big-Big Picture

- However, in type theory, inductive and coinductive types are not well understood.
  - Coq is not type safe [Giménez:1997].
  - Agda does not allow mixing them.

The Big-Big Picture

- How can we fix these problems?
  - Solution: Duality in computation.

The Big-Little Picture

- First, we must understand duality in intuitionistic logic.
- Intuitionistic logic with perfect duality is called bi-intuitionistic logic.
  - **Problem: Categorical model is not well understood [Crolard:2001].**

Bi-intuitionistic Logic

- Classical logic is rich with duality.
  - Even implication has a dual:
    - Subtraction: $A \wedge \neg B = \neg(A \Rightarrow B)$

Bi-intuitionistic Logic

- Intuitionistic subtraction was first studied by [Rauszer: 1974,1977].
- In CS Crolard was the first to introduce the use of subtraction in subtractive logic [Crolard:2001].
  - Application: Constructive coroutines [Crolard:2004].
- In LL, Lambek introduced subtraction in Bilinear Logic [Lambek:1993,Lambek:1995].

Semi-Bilinear Intuitionistic Logic

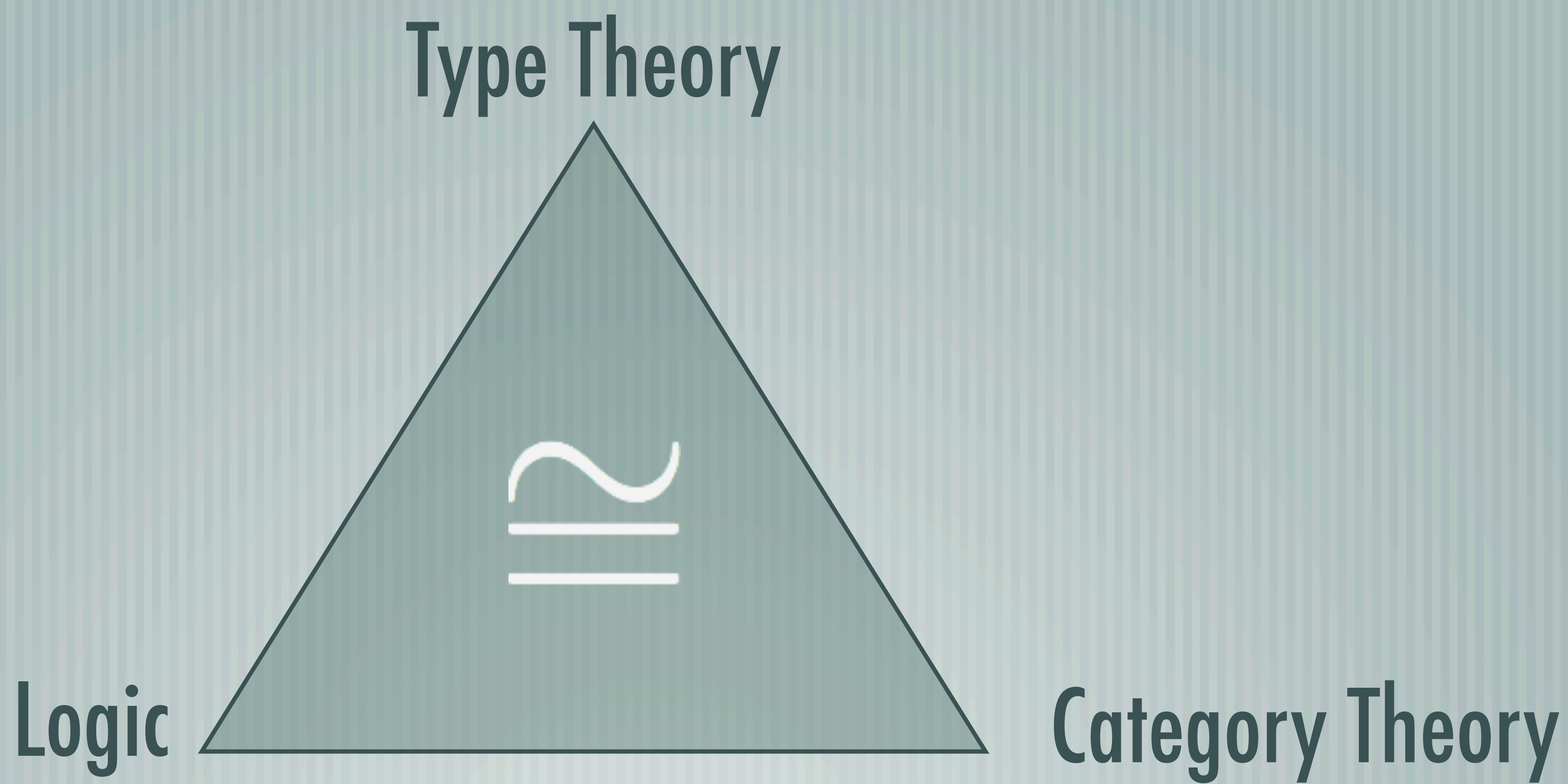
$$
\begin{array}{llll}
\text{(Formulas)} & A, B, C & ::= & \perp \mid \top \mid 1 \mid 0 \mid A \multimap B \mid A \bullet\!\!- B \\
& & & \mid A \otimes B \mid A \oplus B \mid A \times B \\
& & & \mid A + B \mid\, !A \mid\, ?A \\
\text{(Contexts)} & \Gamma, \Delta & ::= & \cdot \mid A \mid \Gamma, \Gamma'
\end{array}
$$

$$
\frac{\begin{array}{c}\Gamma \vdash A, \Delta \\ \Gamma', B \vdash \Delta'\end{array}}{\Gamma, \Gamma', A \multimap B \vdash \Delta, \Delta'}\ \textsc{imp}\mathrm{L}
\qquad
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B, \Delta}\ \textsc{imp}\mathrm{R}
$$

$$
\frac{A \vdash B, \Delta}{\Gamma, A \bullet\!\!- B \vdash \Delta}\ \textsc{sub}\mathrm{L}
\qquad
\frac{\begin{array}{c}\Gamma' \vdash A, \Delta' \\ \Gamma, B \vdash \Delta\end{array}}{\Gamma, \Gamma' \vdash A \bullet\!\!- B, \Delta, \Delta'}\ \textsc{sub}\mathrm{R}
$$

Categorical Investigation

The Three Perspectives of Computation



Subtractive Logic

- In [Crolard:2001] Crolard showed that the categorical model using bi-[CCC]s for subtractive logic is degenerative.
  - There is at most one morphism between any two objects.
- However, this collapse does not occur for categorical models of linear logic.

Categorical Model

- Monoidal Category:

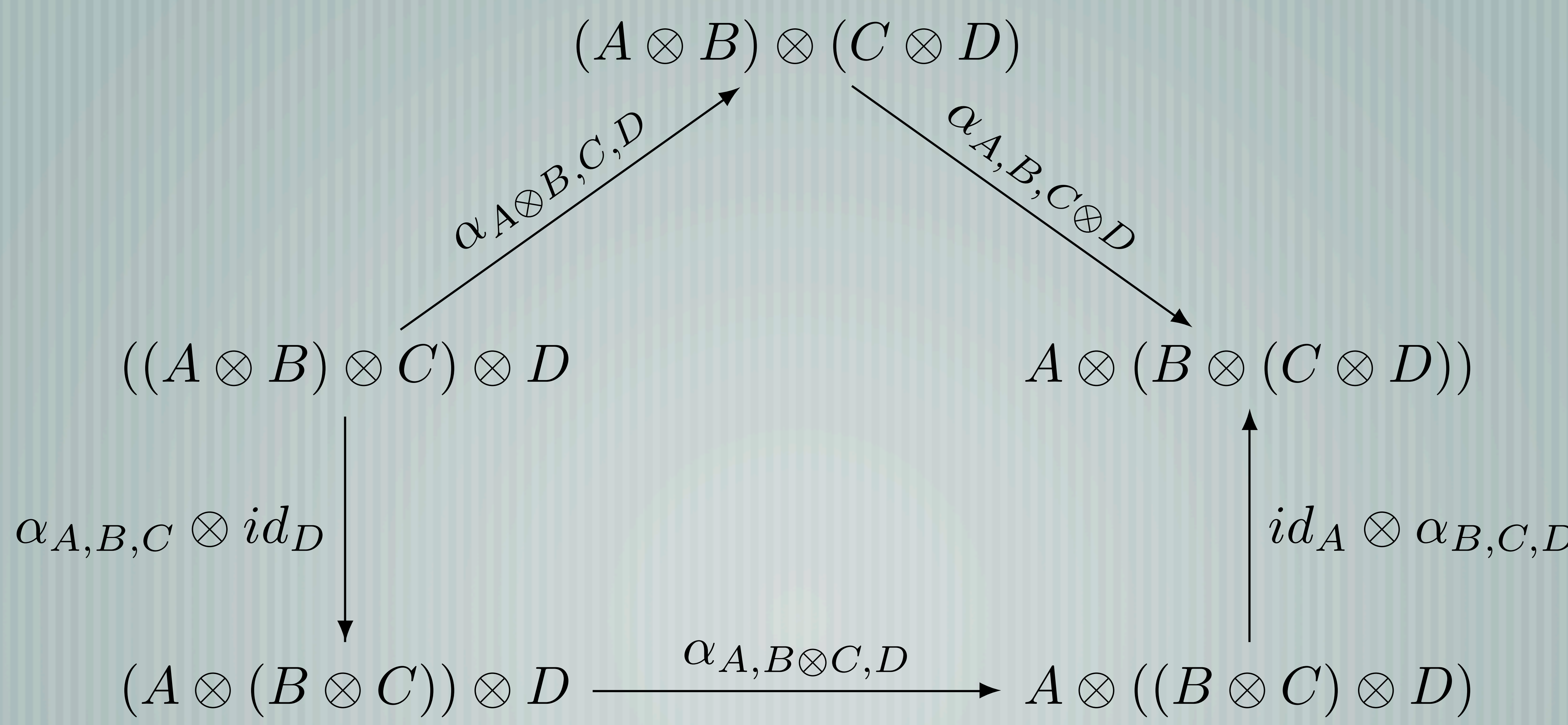
$$(\mathbb{C}, \otimes, T, \alpha_{A,B,C}, \lambda_A, \rho_A)$$

$$\mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C} \qquad (A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C)$$

$$T \otimes A \xrightarrow{\lambda_A} A \qquad A \otimes T \xrightarrow{\rho_A} A$$

Categorical Model

- Monoidal Category:



Categorical Model

- Monoidal Category:

$$(A \otimes T) \otimes B \xrightarrow{\alpha_{A,T,B}} A \otimes (T \otimes B)$$

$$(A \otimes T) \otimes B \xrightarrow{\rho_A \oplus id_B} A \otimes B \qquad A \otimes (T \otimes B) \xrightarrow{id_A \oplus \lambda_B} A \otimes B$$

$$\lambda_T = \rho_T : T \otimes T \to T$$

Categorical Model

Symmetric Monoidal Category:

$$A \otimes B \xrightarrow{\beta_{A,B}} B \otimes A \qquad \beta_{B,A} \circ \beta_{A,B} = id_{A \otimes B}$$

$$\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{\alpha_{A,B,C}} & A \otimes (B \otimes C) & \xrightarrow{\beta_{A,B \otimes C}} & (B \otimes C) \otimes A \\
\Big\downarrow{\scriptstyle \beta_{A,B} \otimes id_C} & & & & \Big\downarrow{\scriptstyle \alpha_{B,C,A}} \\
(B \otimes A) \otimes C & \xrightarrow{\alpha_{B,A,C}} & B \otimes (A \otimes C) & \xrightarrow{id_B \otimes \beta_{A,C}} & B \otimes (C \otimes A)
\end{array}$$

Categorical Model

- Symmetric Monoidal Category:

$\beta_{T,T} : T \otimes T \to T \otimes T$ $\qquad$ $id_{T\otimes T} : T \otimes T \to T \otimes T$

Categorical Model

- Symmetric Monoidal Category:

$$\beta_{T,T} : T \otimes T \to T \otimes T \qquad id_{T \otimes T} : T \otimes T \to T \otimes T$$

$$\beta_{T,T} \neq id_{T \otimes T}$$

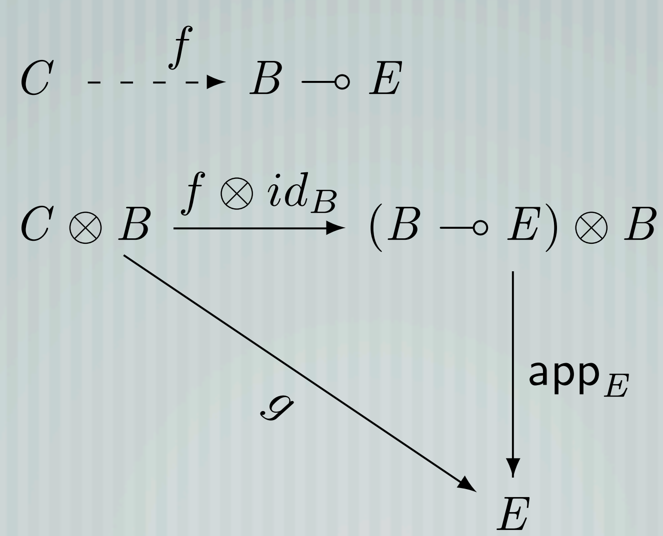
Categorical Model

Closed Symmetric Monoidal Category:

For any object $B \in \mathbb{C}_0$ the functor $- \otimes B : \mathbb{C} \to \mathbb{C}$ has a right adjoint functor $B \multimap - : \mathbb{C} \to \mathbb{C}$. This means that for all objects $A, B, C \in \mathbb{C}_0$, we have the following bijection:

$$\mathbb{C}[A \otimes B, C] \cong \mathbb{C}[A, B \multimap C]$$

that is natural in all arguments. This adjunction implies the following UMP:



Categorical Model

- Bilinearly distributive category:
  - SMCC: $(\mathbb{C}, \otimes, T, \alpha_{A,B,C}, \lambda_A, \rho_A, \multimap)$
  - SMCCC: $(\mathbb{C}, \oplus, I, \tilde{\alpha}_{A,B,C}, \tilde{\lambda}_A, \tilde{\rho}_A, \bullet\!\!-)$
  - Distributive:

$$
\begin{array}{lcl}
\mathsf{dist}^L_L\,A\,B\,C & : & A \otimes (B \oplus C) \to (A \otimes B) \oplus C \\
\mathsf{dist}^R_R\,A\,B\,C & : & (B \oplus C) \otimes A \to B \oplus (C \otimes A) \\
\mathsf{dist}^L_R\,A\,B\,C & : & A \otimes (B \oplus C) \to B \oplus (A \otimes C) \\
\mathsf{dist}^R_L\,A\,B\,C & : & (B \oplus C) \otimes A \to (B \otimes A) \oplus C
\end{array}
$$

Categorical Model

- Start with a symmetric monoidal closed category.
  - Think of linear implication, and tensor.
- However, this model does not handle multiple conclusions.
  - Extend the model to be linearly distributive [Cockett, Seely, Trimble, Blute : 1997, 1999].
  - Add the coclosure, that is, subtraction.
  - Interpret sequents as: $[\![\Gamma \vdash \Delta]\!] = \phi : [\![\Gamma]\!] \to [\![\Delta]\!]$

Categorical Model

Theorem 1. *Assume $\mathbb{C}$ is an arbitrary bilinearly distributive category. If $\Gamma \vdash \Delta$, then there exists a morphism $f \in \mathbb{C}[[\![\Gamma]\!], [\![\Delta]\!]]$.*

A new idea!

- Subtractive logic has a simple definition, but the Dragalin restriction results in a failure of cut elimination.
  - Counter Example: $A \to^{+} (A \to^{-} A \to^{+} \langle - \rangle) \to^{-} \langle + \rangle$

Bi-intuitionistic Logic

- [Goré:2000] gives a display calculus that is a bi-intuitionistic logic with cut-elimination.
- [Goré et al.:2010] gives a logic using nested sequents which is has cut-elimination.
  - 2-Category?
- [Pinto and Uustalu:2009-2010] give a labeled sequent calculus for bi-intuitionistic logic that has cut-elimination.

Bi-intuitionistic Logic

$$\frac{n' \notin |G| \qquad (G, n \preccurlyeq^p n'); \Gamma, p\ T_1 @ n' \vdash^p_{n'} T_2}{G; \Gamma \vdash^p_n T_1 \rightarrow_p T_2}\ \textsc{imp}$$

$$\frac{G \vdash n \preccurlyeq^*_{\bar{p}} n' \qquad G; \Gamma \vdash^{\bar{p}}_{n'} T_1 \qquad G; \Gamma \vdash^p_{n'} T_2}{G; \Gamma \vdash^p_n T_1 \rightarrow_{\bar{p}} T_2}\ \textsc{impBar}$$

The Categorical Model

A **preordered category** is a tuple $(\mathbb{P}, \mathbb{C}, \otimes, \oplus, \mathsf{I}, \mathsf{T})$ where $\mathbb{P}$ is the base preorder, and $\mathbb{C}$ has the following data:

- a collection of objects denoted $A@w$, where $w \in \mathsf{Obj}(\mathbb{P})$
- a collection of morphisms denoted $f : M; A@w \to B@w'$
- for any preorder M, and object $A@w$, there exists a morphism $\mathsf{id}_{A@w} : M; A@w \to A@w$ called the identity morphism
- for any morphisms $f : M_1; A@w_1 \to B@w_2$, and $g : M_2; B@w_2 \to C@w_3$, there exists a morphism $f; g : M_1, M_2; A@w_1 \to C@w_3$,

The Categorical Model

A **preordered category** is a tuple $(\mathbb{P}, \mathbb{C}, \otimes, \oplus, \mathsf{I}, \mathsf{T})$ where $\mathbb{P}$ is the base preorder, and $\mathbb{C}$ has the following data:

- ____ for any morphisms $f : M_1; A@w_1 \to B@w_2$, and $g \in M_1[w_1, w_3]$, there exists the morphism $f \overset{s}{\leadsto} g : M_1; A@w_3 \to B@w_2$
- ____ for any morphisms $f : M_1; A@w_1 \to B@w_2$, and $g \in M_1[w_2, w_3]$, there exists the morphism $f \overset{t}{\leadsto} g : M_1; A@w_1 \to B@w_3$
- ____ for any morphism $f : M_1; A@w_1 \to B@w_2$, $\mathsf{id}_{A@w_1}; f = f$, and $f; \mathsf{id}_{B@w_2} = f$
- ____ for any morphisms $f : M_1; A@w_1 \to B@w_2$, and $g : M_2; B@w_2 \to C@w_3$, $h : M_3; C@w_3 \to D@w_4$, $f; (g; h) = (f; g); h$
- ____ all finite products, denoted by $\otimes$, and coproducts, denoted $\oplus$

Conclusion

- Key Points:
  - duality can be exploited to solve interesting problems,
  - category theory is a powerful tool,
  - SBILL is a new ILL with perfect duality with
  - a categorical model.

Thank you!