Dualized Type Theory

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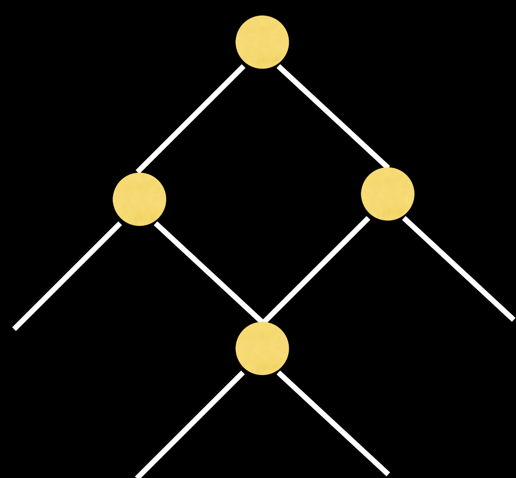
GEORGIA REGENTS UNIVERSITY AUGUSTA

THE UNIVERSITY OF IOWA

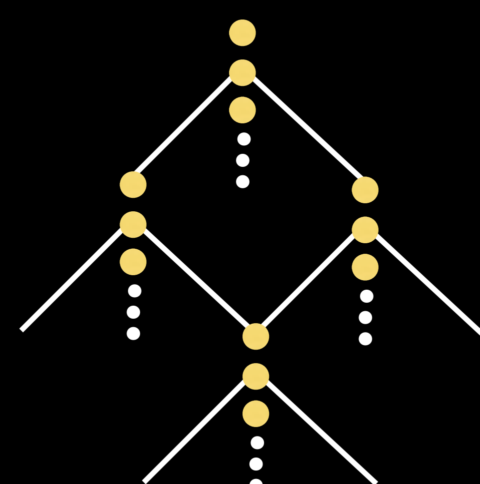
Long-term Goal

Inductive Data:



Coinductive Data:

Mixed Ind-Coind Data:



- Coq is not type safe [Giménez:1997].
- P. Selinger (2003): Some Remarks on Control Categories. Manuscript.

Bi-intuitionistic (BINT) Logic

- Intuitionistic logic with perfect duality.
  - The dual of implication is subtraction or exclusion.
    - First studied by the Cycilia Rauszer in the 70's.

BINT Logic and Type Theory

- Symmetric Comb. Logic: Filinski.
- Subtractive logic: Crolard.
- Logic for Pragmatics: Bellin, and Biasi and Aschieri.
- Nested Sequents: Rajeev Goré , Linda Postniece & Alwen Tiu.
- Labled BINT: Pinto and Uustalu.

We really want to stick to sequent calculi.

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Labeled BINT

LK+Subtraction:

$$A_1, \ldots, A_j \vdash B_1, \ldots, B_k$$

Labeled BINT:

$$n_1 : A_1, \ldots, n_j : A_j \vdash_G m_1 : B_1, \ldots, m_k : B_k$$

- 2009 : Proof Search and Counter-Model Construction for Bi-intuitionistic Propositional Logic with Labelled Sequents.

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Labeled BINT

LK+

A

Labe

n_1

on

$$\frac{\begin{array}{c} n' \notin |G|, |\Gamma|, |\Delta| \\ \Gamma, n' : T_1 \vdash_{G \cup \{(n,n')\}} n' : T_2, \Delta \end{array}}{\Gamma \vdash_G n : T_1 \supset T_2, \Delta}\ \textsc{ImpR}$$

$$\frac{\begin{array}{c} n' G n \\ \Gamma \vdash_G n' : T_1, \Delta \\ \Gamma, n' : T_2 \vdash_G \Delta \end{array}}{\Gamma \vdash_G n : T_1 \prec T_2, \Delta}\ \textsc{SubR}$$

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Labeled BINT

$$\frac{\Gamma \vdash_{G \cup \{(n,n)\}} \Delta}{\Gamma \vdash_G \Delta} \text{ REFL}$$

$$\frac{n_1 G n_2 \quad n_2 G n_3 \quad \Gamma \vdash_{G \cup \{(n_1,n_3)\}} \Delta}{\Gamma \vdash_G \Delta} \text{ TRANS}$$

$$\frac{nGn' \quad \Gamma, n : T, n' : T \vdash_G \Delta}{\Gamma, n : T \vdash_G \Delta} \text{ MonoL}$$

$$\frac{n'Gn \quad \Gamma \vdash_G n' : T, n : T, \Delta}{\Gamma \vdash_G n : T, \Delta} \text{ MonoR}$$

Dualized Intuitionistic Logic (DIL)

- A simplification of labeled BINT.
  - A new dualized syntax.
  - Pushed the refl and trans rules to the leaves.
  - Removed the monotonicity rules.

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Dualized Intuitionistic Logic (DIL)

$$\frac{G \vdash n \preccurlyeq_p^* n'}{G; \Gamma, p\ A \mathbin{@} n, \Gamma' \vdash p\ A \mathbin{@} n'}\ \textsc{ax} \qquad \frac{}{G; \Gamma \vdash p\ \langle p \rangle \mathbin{@} n}\ \textsc{unit}$$

$$\frac{G; \Gamma \vdash p\ A \mathbin{@} n \qquad G; \Gamma \vdash p\ B \mathbin{@} n}{G; \Gamma \vdash p\ (A \wedge_p B) \mathbin{@} n}\ \textsc{and}$$

$$\frac{G; \Gamma \vdash p\ A_d \mathbin{@} n}{G; \Gamma \vdash p\ (A_1 \wedge_{\bar{p}} A_2) \mathbin{@} n}\ \textsc{andBar}$$

$$\frac{\begin{array}{l} n' \notin |G|, |\Gamma| \\ (G, n \preccurlyeq^p n'); \Gamma, p\ A \mathbin{@} n' \vdash p\ B \mathbin{@} n' \end{array}}{G; \Gamma \vdash p\ (A \rightarrow_p B) \mathbin{@} n}\ \textsc{imp}$$

$$\frac{\begin{array}{l} G \vdash n \preccurlyeq_{\bar{p}}^* n' \\ G; \Gamma \vdash \bar{p}\ A \mathbin{@} n' \qquad G; \Gamma \vdash p\ B \mathbin{@} n' \end{array}}{G; \Gamma \vdash p\ (A \rightarrow_{\bar{p}} B) \mathbin{@} n}\ \textsc{impBar}$$

$$\frac{G; \Gamma, \bar{p}\ A \mathbin{@} n \vdash +\ B \mathbin{@} n' \qquad G; \Gamma, \bar{p}\ A \mathbin{@} n \vdash -\ B \mathbin{@} n'}{G; \Gamma \vdash p\ A \mathbin{@} n}\ \textsc{cut}$$

Dualized Intuitionistic Logic (DIL)

Labeled BINT:

$$n_1 : A_1, \ldots, n_j : A_j \vdash_G m_1 : B_1, \ldots, m_k : B_k$$

DIL:

$$G; +\, A_1 \mathbin{@} n_1, \ldots, +\, A_j \mathbin{@} n_j, -\, B_2 \mathbin{@} m_2, \ldots, -\, B_k \mathbin{@} m_k \vdash +\, B_1 \mathbin{@} m_1$$

$$G; p_1\, A_1 \mathbin{@} n_1, \ldots, p_2\, A \mathbin{@} n_2 \vdash p\, B \mathbin{@} n$$

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$G; p_1\ A_1 \mathbin{@} n_1, \ldots, p_2\ A \mathbin{@} n_2 \vdash p\ B \mathbin{@} n$

Dualized Intuitionistic Logic (DIL)

Labe

$n_1 :$

DIL:

$G; +$ … m_1

$G; p$

$$\frac{n' \notin |G|, |\Gamma| \qquad (G, n \preccurlyeq_p n'); \Gamma, p\, A \mathbin{@} n' \vdash p\, B \mathbin{@} n'}{G; \Gamma \vdash p\,(A \to_p B) \mathbin{@} n}\ \textsc{imp}$$

$$\frac{G \vdash n \preccurlyeq^*_{\bar{p}} n' \qquad G; \Gamma \vdash \bar{p}\, A \mathbin{@} n' \qquad G; \Gamma \vdash p\, B \mathbin{@} n'}{G; \Gamma \vdash p\,(A \to_{\bar{p}} B) \mathbin{@} n}\ \textsc{impBar}$$

Consistency and Completeness

- Consistency was proven (in Agda) w.r.t. the following notion of validity*:

$$[\![G; \Gamma \vdash p\ A\ @\ n]\!]_N = \text{ if } [\![G]\!]_N \text{ and } [\![\Gamma]\!]_N, \text{ then } p[\![A]\!]_{(N\ n)}$$

- Completeness is shown by reduction to L:

If $\ulcorner G \urcorner; \Gamma' \vdash +\, A\ @\ n$ is an activation of the derivable L-sequent $\Gamma \vdash_G \Delta$, then $\ulcorner G \urcorner; \Gamma' \vdash +\, A\ @\ n$ is derivable.

* https://github.com/heades/DIL-consistency

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Consistency and Completeness

$$\frac{G;\Gamma,\bar{p}\ A\ @\ n \vdash +\ B\ @\ n' \qquad G;\Gamma,\bar{p}\ A\ @\ n \vdash -\ B\ @\ n'}{G;\Gamma \vdash p\ A\ @\ n}\ \textsc{cut}$$

$\Rightarrow$

$$\frac{p\ B\ @\ n' \in (\Gamma,\bar{p}\ A\ @\ n) \qquad G;\Gamma,\bar{p}\ A\ @\ n \vdash \bar{p}\ B\ @\ n'}{G;\Gamma \vdash p\ A\ @\ n}\ \textsc{axCut}$$

$$\frac{\bar{p}\ B\ @\ n' \in (\Gamma,\bar{p}\ A\ @\ n) \qquad G;\Gamma,\bar{p}\ A\ @\ n \vdash p\ B\ @\ n'}{G;\Gamma \vdash p\ A\ @\ n}\ \textsc{axCutBar}$$

Dualized Type Theory

$$\frac{G \vdash n \preccurlyeq^*_p n'}{G; \Gamma, x : p\ A\ @\ n, \Gamma' \vdash x : p\ A\ @\ n'}\ \textsc{Ax} \qquad \frac{}{G; \Gamma \vdash \mathbf{triv} : p\ \langle p \rangle\ @\ n}\ \textsc{Unit}$$

$$\frac{G; \Gamma \vdash t_1 : p\ A\ @\ n \qquad G; \Gamma \vdash t_2 : p\ B\ @\ n}{G; \Gamma \vdash (t_1, t_2) : p\ (A \wedge_p B)\ @\ n}\ \textsc{And}$$

$$\frac{G; \Gamma \vdash t : p\ A_d\ @\ n}{G; \Gamma \vdash \mathbf{in}_d\ t : p\ (A_1 \wedge_{\bar{p}} A_2)\ @\ n}\ \textsc{AndBar}$$

$$\frac{\begin{array}{l} n' \notin |G|, |\Gamma| \\ (G, n \preccurlyeq^p n'); \Gamma, x : p\ A\ @\ n' \vdash t : p\ B\ @\ n' \end{array}}{G; \Gamma \vdash \lambda x.t : p\ (A \rightarrow_p B)\ @\ n}\ \textsc{Imp}$$

$$\frac{\begin{array}{l} G \vdash n \preccurlyeq^*_{\bar{p}} n' \\ G; \Gamma \vdash t_1 : \bar{p}\ A\ @\ n' \qquad G; \Gamma \vdash t_2 : p\ B\ @\ n' \end{array}}{G; \Gamma \vdash \langle t_1, t_2 \rangle : p\ (A \rightarrow_{\bar{p}} B)\ @\ n}\ \textsc{ImpBar}$$

$$\frac{\begin{array}{c} G; \Gamma, x : \bar{p}\ A\ @\ n \vdash t_1 : +\ B\ @\ n' \\ G; \Gamma, x : \bar{p}\ A\ @\ n \vdash t_2 : -\ B\ @\ n' \end{array}}{G; \Gamma \vdash \nu\, x.t_1 \bullet t_2 : p\ A\ @\ n}\ \textsc{Cut}$$

Dualized Type Theory

$$\Gamma' =^{\mathrm{def}} \Gamma, y : -\,B \mathbin{@} n$$

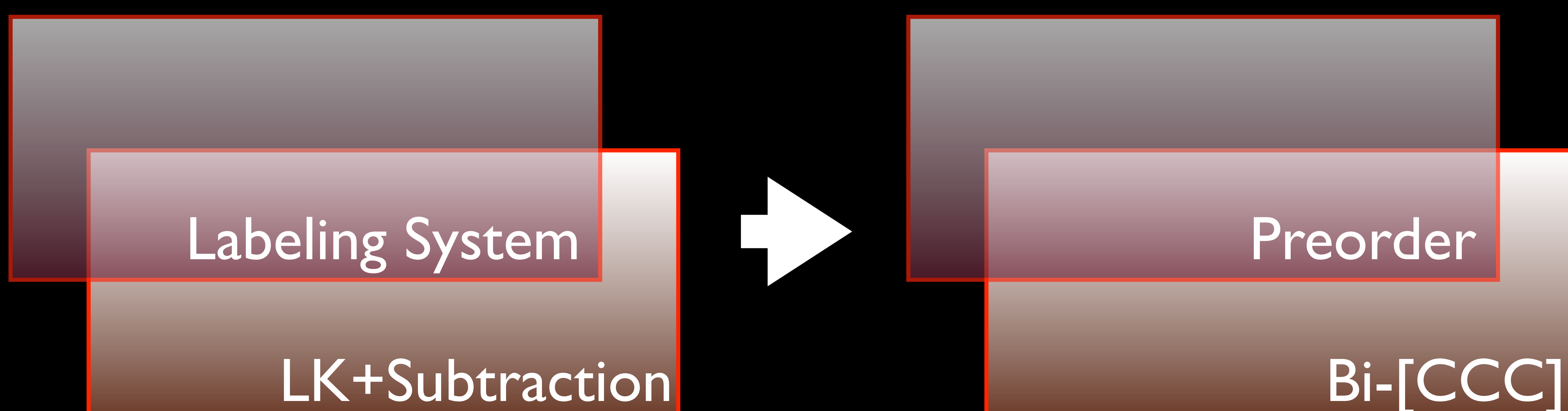
$$\dfrac{\begin{array}{l} G;\Gamma \vdash \lambda x.t : +\,(A \to_+ B) \mathbin{@} n \\ G;\Gamma \vdash t' : +\,A \mathbin{@} n \end{array} \qquad \dfrac{\dfrac{}{G;\Gamma' \vdash y : -\,B \mathbin{@} n}\ \textsc{ax}}{G;\Gamma' \vdash \langle t', y\rangle : -\,(A \to_+ B) \mathbin{@} n}\ \textsc{impBar}}{G;\Gamma \vdash \nu\, y.\lambda x.t \bullet \langle t', y\rangle : +\,B \mathbin{@} n}\ \textsc{cut}$$

Dualized Type Theory

- Metatheory of DTT:
  - Type preservation.
  - Strong normalization.
    - By forgetting the labels and proving SN of LK+subtraction using classical realizability.

Plans for the Future

- Add inductive and coinductive types.
  - Need a categorical model of DTT.
    - Preordered Categories:



Plans for the Future

Category: $(\mathcal{P}, \mathcal{C})$

Objects: $A@n_1, B@n_2, \cdots$

Morphisms: $A_1@n_1, A_2@n_2, \ldots, A_i@n_i \xrightarrow{f^M} B@n$

DIL-sequent:

$$G; +A_1@n_1, \ldots, +A_2@n_2 \vdash +B@n$$

$$[\![A_1]\!]@[\![n_1]\!], \ldots, [\![A_2]\!]@[\![n_2]\!] \xrightarrow{f^{[\![G]\!]}} [\![B]\!]@[\![n]\!]$$

Thank you!