Multiple Conclusion Linear Logic: Cut-elimination and more

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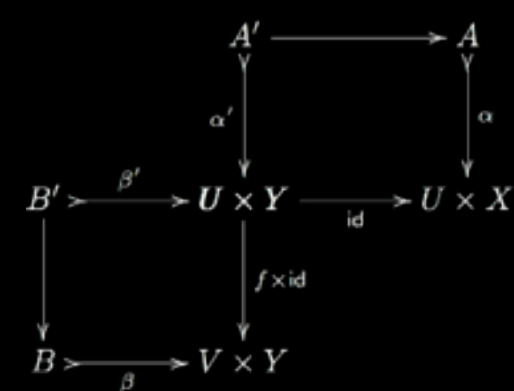
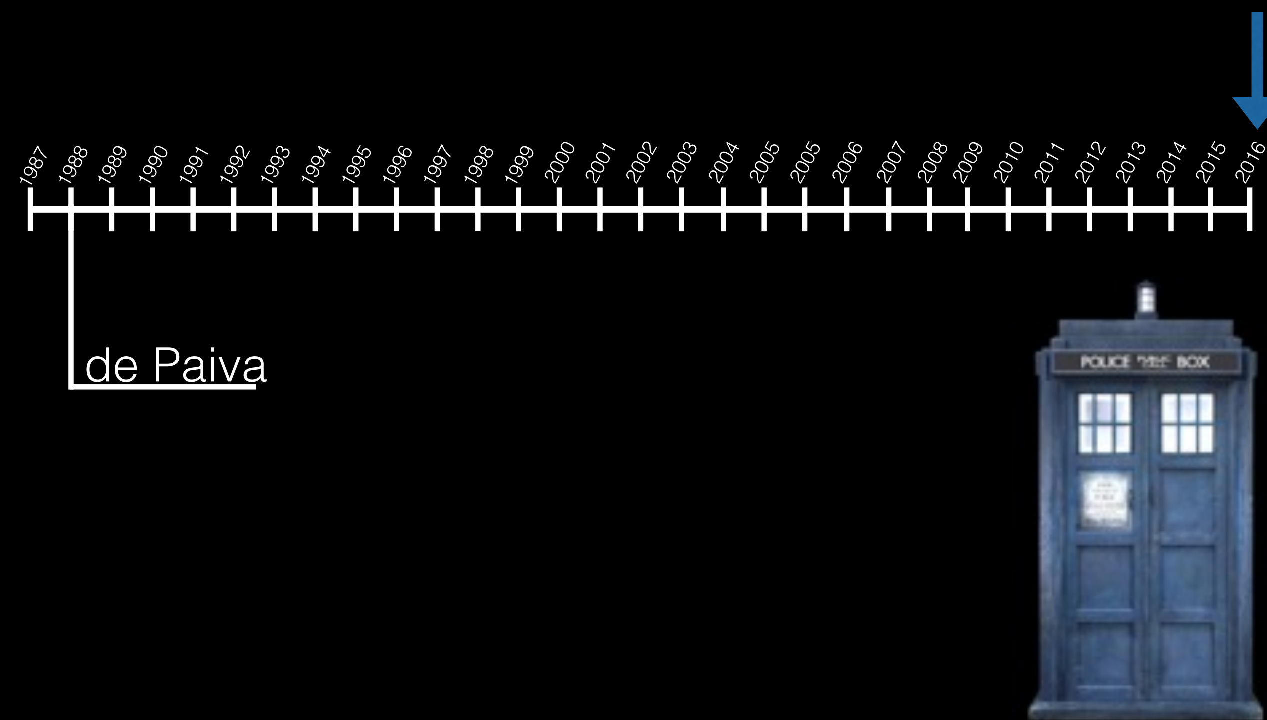


LFCS 2015

Full Intuitionistic linear Logic (FILL): Cut Elimination

Full Intuitionistic Linear Logic



(1988) FILL: de Paiva

Dialectica Categories:

Came from de Paiva’s thesis work on a categorical interpretation of Godel’s Dialectica interpretation.

(1988) FILL: de Paiva

Dialectica Categories:

One of the first well-behaved categorical models of intuitionistic linear logic.

(1988) FILL: de Paiva

Dialectica Categories:

- Symmetric Monoidal Closed:
  - Multiplicative Conjunction (tensor)
  - Linear Implication

(1988) FILL: de Paiva

Dialectica Categories:

- Multiplicative Disjunction (par)
- Tensor Distributes over par

(1988) FILL: de Paiva

Dialectica Categories:

- Products and Coproducts:
  - Additive conjunction and disjunction
- Linear modality: of-course (!)

(1988) FILL: de Paiva

Full Intuitionistic Linear Logic (FILL):

- Multiplicatives: Tensor and Par
  - Tensor and Par distribute
- Additives: Conjunction and Disjunction
- Linear Implication
- Linear Exponential: of-course (!)

(1988) FILL: de Paiva

Classical Linear Logic:

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \parr B \vdash \Delta}\ \textsc{par}\mathrm{L} \qquad \frac{\Gamma \vdash A \mid B \mid \Delta}{\Gamma \vdash A \parr B \mid \Delta}\ \textsc{par}\mathrm{R}$$

$$\frac{\Gamma \vdash A \mid \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, A \multimap B, \Gamma' \vdash \Delta, \Delta'}\ \textsc{imp}\mathrm{L} \qquad \frac{\Gamma, A \vdash B \mid \Delta}{\Gamma \vdash A \multimap B \mid \Delta}\ \textsc{imp}\mathrm{R}$$

Show tensor and par

(1988) FILL: de Paiva

Full Intuitionistic Linear Logic:

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \parr B \vdash \Delta}\ \textsc{parL} \qquad \frac{\Gamma \vdash A \mid B \mid \Delta}{\Gamma \vdash A \parr B \mid \Delta}\ \textsc{parR}$$

$$\frac{\Gamma \vdash A \mid \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, A \multimap B, \Gamma' \vdash \Delta, \Delta'}\ \textsc{impL} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B \mid \Delta}\ \textsc{impR}$$

(1988) FILL: de Paiva

Full Intuitionistic Linear Logic:

Failure of cut-elimination (Schellinx:1991).

FILL: Term Assignment

(1993) FILL: Hyland and de Paiva

Does not depend on

$$\frac{\Gamma \vdash A \mid \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, A \multimap B, \Gamma' \vdash \Delta, \Delta'} \textsc{impL} \qquad\qquad \frac{\Gamma, A \vdash B \mid \Delta}{\Gamma \vdash A \multimap B \mid \Delta} \textsc{impR}$$

This is called the FILL condition

(1993) FILL: Hyland and de Paiva

Term Assignment:

$$
\begin{aligned}
\text{(Terms)} \qquad t, e ::= {} & x \\
& \mid * \mid \circ \\
& \mid t_1 \otimes t_2 \mid t_1 \parr t_2 \mid \mathsf{let}\ t\ \mathsf{be}\ p\ \mathsf{in}\ e \\
& \mid \lambda x.t \mid t_1\ t_2
\end{aligned}
$$

$$
\text{(Patterns)} \qquad p ::= * \mid - \mid x \mid p_1 \otimes p_2 \mid p_1 \parr p_2
$$

(1993) FILL: Hyland and de Paiva

Term Assignment:

$$\frac{\Gamma, x : A \vdash \Delta \quad \Gamma', y : B \vdash \Delta'}{\Gamma, \Gamma', z : A \parr B \vdash \mathsf{let}\ z\ \mathsf{be}\ (x \parr -)\ \mathsf{in}\ \Delta \mid \mathsf{let}\ z\ \mathsf{be}\ (- \parr y)\ \mathsf{in}\ \Delta'} \ \textsc{ParL}$$

$$\frac{\Gamma \vdash \Delta \mid e : A \mid f : B \mid \Delta'}{\Gamma \vdash \Delta \mid e \parr f : A \parr B \mid \Delta'} \ \textsc{ParR}$$

(1993) FILL: Hyland and de Paiva

Term Assignment:

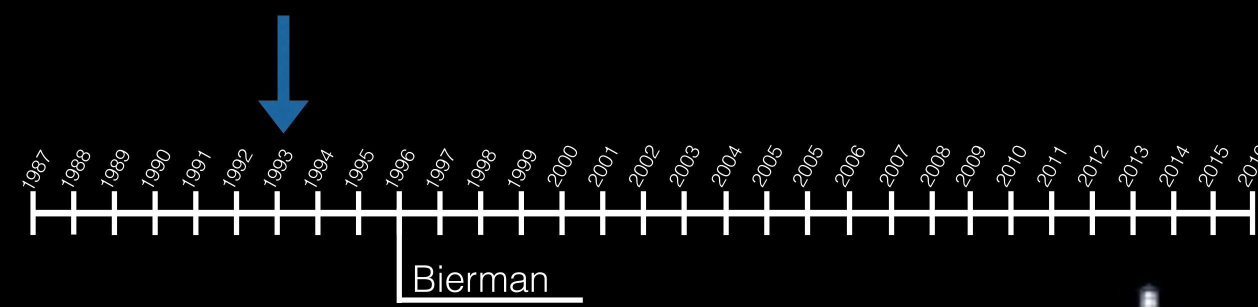
$$\frac{\Gamma \vdash e : A \mid \Delta \quad \Gamma', x : B \vdash \Delta'}{\Gamma, y : A \multimap B, \Gamma' \vdash \Delta \mid [y\ e/x]\Delta'}\ \textsc{ImpL}$$

$$\frac{\Gamma, x : A \vdash e : B \mid \Delta \quad x \notin \mathsf{FV}(\Delta)}{\Gamma \vdash \lambda x.e : A \multimap B \mid \Delta}\ \textsc{ImpR}$$

(1993) FILL: Hyland and de Paiva

But, there is a problem with this version.

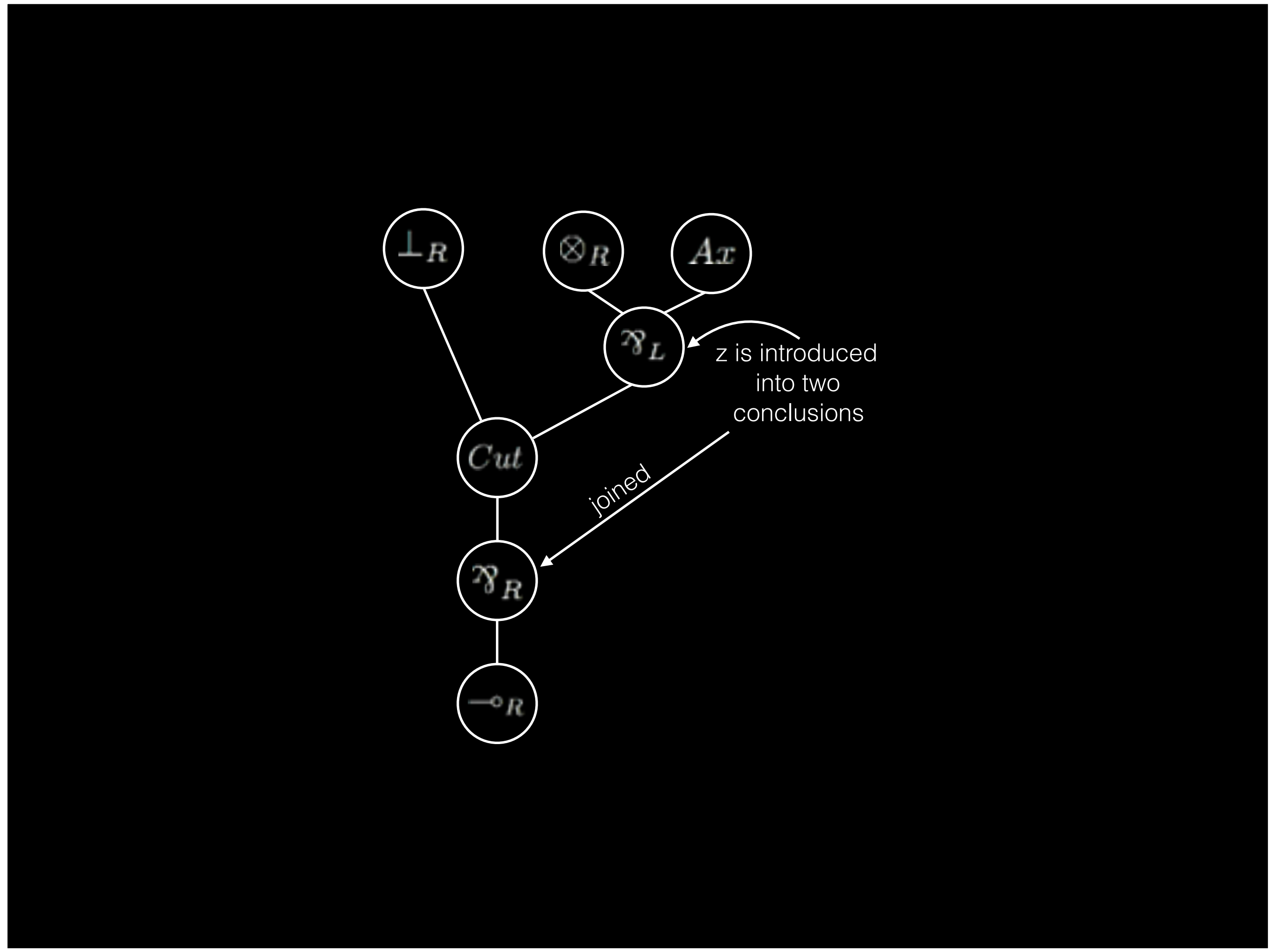
FILL: Another Counterexample



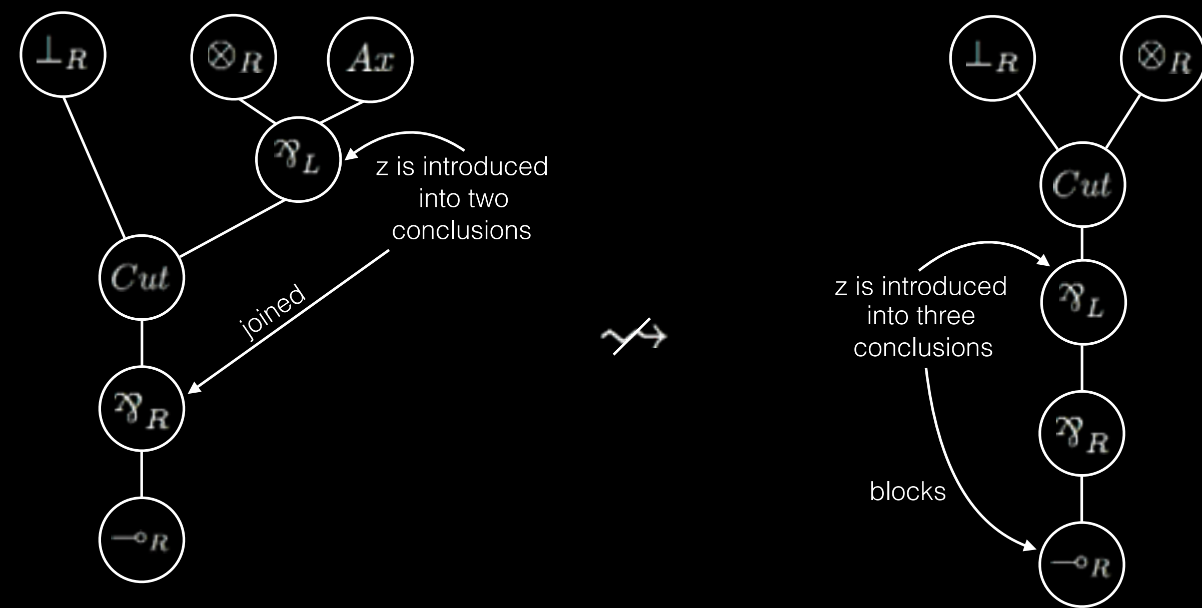
(1996) FILL: Bierman

Cut-elimination still fails!

$$\frac{\Gamma, x : A \vdash \Delta \quad \Gamma', y : B \vdash \Delta'}{\Gamma, \Gamma', z : A \parr B \vdash \mathsf{let}\ z\ \mathsf{be}\ (x \parr -)\ \mathsf{in}\ \Delta \mid \mathsf{let}\ z\ \mathsf{be}\ (- \parr y)\ \mathsf{in}\ \Delta'}\ \textsc{ParL}$$



$\bot_R$
$\otimes_R$
Ax
$⅋_L$
z is introduced into two conclusions
Cut
joined
$⅋_R$
$\multimap_R$



See “A note on full intuitionistic linear logic” for the complete counterexample.

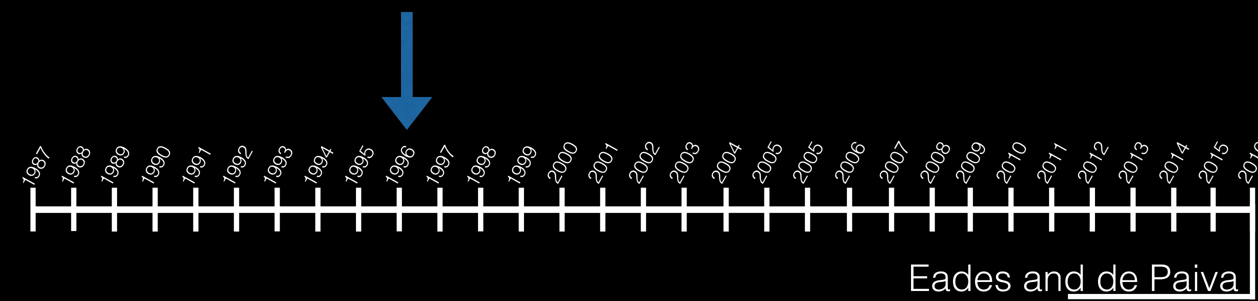
(1996) FILL: Bierman

Bellin proposed the following rule:

$$\frac{\Gamma, x : A \vdash \Delta \quad \Gamma', y : B \vdash \Delta'}{\Gamma, \Gamma', z : A \parr B \vdash \text{let-pat}\, z\, (x \parr -)\, \Delta \mid \text{let-pat}\, z\, (- \parr y)\, \Delta'} \quad \textsc{ParL}$$

In Bellin:1997 he shows cut-elimination using proof nets.

FILL: A Direct Proof



(2016) FILL: Eades and de Paiva

If $\Gamma \vdash t_1 : A_1, \ldots, t_i : A_i$ steps to $\Gamma \vdash t'_1 : A_1, \ldots, t'_i : A_i$ using the cut-elimination procedure, then $t_j \rightsquigarrow^* t'_j$ for $1 \leq j \leq i$.

We adopt Bellin's rule and gave a direct proof of cut-elimination.
The proof holds by a straightforward adaption of the cut-elimination proof for classical linear logic.

Dialectica Categories

Basic Dialectica Categories

$\mathsf{Dial}_2(\mathsf{Sets})$

Objects:

(U, X, α), where U and X are sets, and $\alpha \subseteq U \times X$

Basic Dialectica Categories

$\mathsf{Dial}_2(\mathsf{Sets})$

Morphisms:

$(f, F) : (U, X, \alpha) \longrightarrow (V, Y, \beta)$, where

- $f : U \longrightarrow V$ and $F : Y \longrightarrow X$
- For any $u \in U$ and $y \in Y$, $\alpha(u, F(y))$ implies $\beta(f(u), y)$

DC: F : U x Y -> X

- Model of INT
- Weak co-products
- No par

Dialectica Categories,
Linear Categories,
and
LNL Models

Discuss the models

Linear Categories (Bierman:1994)

- Symmetric Monoidal Closed Category: $(\mathcal{L}, I, \otimes, \multimap)$
- Monoidal Comonad: $(\mathcal{L}, !A, e_A, d_A)$

e_A: Weakening
d_A: Contraction

The distributors must satisfy several coherence conditions which can all be found in Cockett:1997

Full Linear Categories

Linear Category

- Symmetric Monoidal Closed Category: $(\mathcal{L}, I, \otimes, \multimap)$
- Monoidal Comonad: $(\mathcal{L}, !A, e_A, d_A)$

Symmetric Monoidal Structure: $(\mathcal{L}, \perp, \parr)$

Distributors:

$\mathsf{dist}_1 : A \otimes (B \parr C) \to (A \otimes B) \parr C$

$\mathsf{dist}_2 : (A \parr B) \otimes C \to A \parr (B \otimes C)$

e_A: Weakening

d_A: Contraction

The distributors must satisfy several coherence conditions which can all be found in Cockett:1997

LNL Model (Benton:1996)

- Cartesian Closed Category: $(\mathcal{C}, 1, \times, \rightarrow)$
- Symmetric Monoidal Closed Category: $(\mathcal{L}, I, \otimes, \multimap)$
- Symmetric Monoidal Adjunction: $\mathcal{C} : (F, m) \vdash (G, n) : \mathcal{L}$

The distributors must satisfy several coherence conditions which can all be found in Cockett:1997

Full LNL Model

LNL Model:

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The distributors must satisfy several coherence conditions which can all be found in Cockett:1997

Dialectica Categories are Full LNL Models

The category $\mathsf{Dial}_2(\mathsf{Sets})$ is a full linear category.*

New: ! must be sym. monoidal.

(Section 2.2.1 of Benton:1994) Every LNL model is a linear category.

(Section 2.2.2 of Benton:1994) Every linear category is a LNL model.

*Formalized in Agda

The point of these calculations is to show that the several different axiomatizations available for models for linear logic are consistent and that a model proved sound and complete according to Seely's definition (using the Seely isomorphisms $!(A \times B) = !A \otimes !B$ and $!1 = \top$ but adding to it monoidicicty of the comonad) is indeed sound and complete as a LNL model too.

Dialectica Categories
and
Tensorial Logic

Full Tensorial Logic (Melliès:2009)

Symmetric Monoidal Category: $(\mathcal{L}, I, \otimes)$

Tensorial Logic: 2007 Jean-Yves Girard on the occasion of his 60th birthday.

Resource modalities in tensorial logic, Joint work with Nicolas Tabareau, Annals of Pure and Applied Logic (2009), Volume 161, Issue 5, Pages 632-653.

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Family of bijections:

$$\phi_{A,B,C} : \mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \neg(B \otimes C))$$

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Cartesian Category: $(\mathcal{C}, 1, \times)$

Exponential Resource Modality: $\mathcal{C} : F \dashv G : \mathcal{L}$

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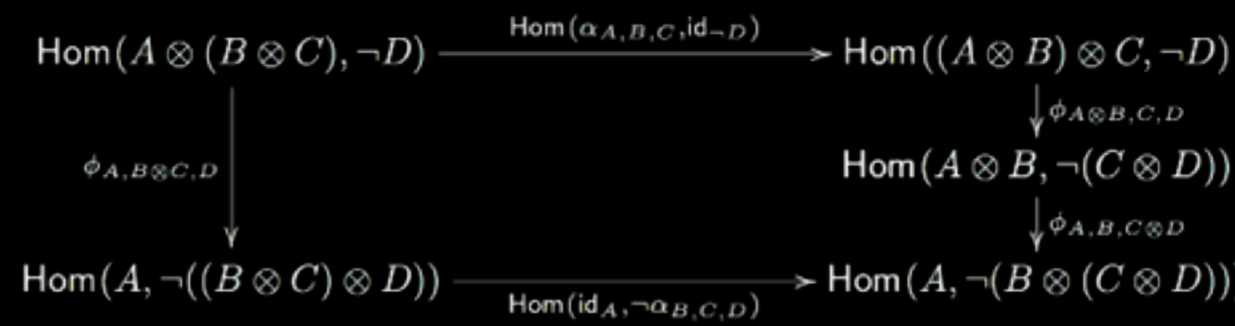

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Dialectica Categories?

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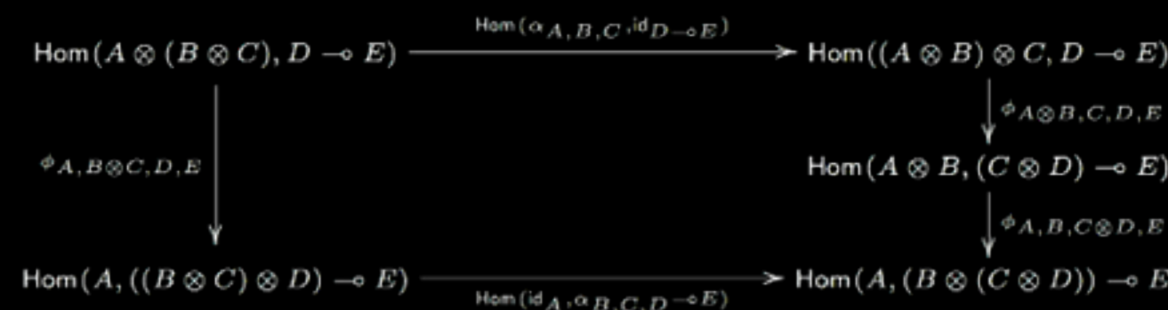
Dialectica Categories Model Tensorial Logic

In any monoidal closed category, $\mathcal{C}$, there is a natural bijection

$$\phi_{A,B,C,D} : \mathsf{Hom}_{\mathcal{C}}(A \otimes B, C \multimap D) \cong \mathsf{Hom}_{\mathcal{C}}(A, (B \otimes C) \multimap D)$$

Furthermore, the following diagram commutes:



To get tensorial negation:

Replace D in \phi with \perp

Replace E in the diagram with \perp

Dialectica Categories?

✓Symmetric Monoidal Category: $(\mathcal{L}, I, \otimes)$

✓Tensorial Negation: $\neg : \mathcal{L} \longrightarrow \mathcal{L}^{\mathsf{op}}$

✓Family of bijections:

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Dialectica Categories Model Tensorial Logic

The category $\mathsf{Dial}_2(\mathsf{Sets})$ is a model of full tensorial logic.

- Construct the co-Kleisli category $\mathsf{Dial}_2(\mathsf{Sets})_!$
- $\mathsf{Dial}_2(\mathsf{Sets}) : F \dashv G : \mathsf{Dial}_2(\mathsf{Sets})_!$: $\mathsf{Dial}_2(\mathsf{Sets})$ is a full LNL model.
- Finally, show that $\mathsf{Dial}_2(\mathsf{Sets})_!$ is cartesian.

Formalized in Agda

Future Work

Concurrency

- Binary Session Types (Honda et al.:1998)
  - A proof theory in intuitionistic linear logic (Caires and Pfenning:2010)

Concurrency

- Multiparty Session Types (Honda et al.:2008)
  - A proof theory in classical linear logic (Carbone et al:2015)
  - Requires Multiple Conclusions
  - Capitalizes on classical duality.

Concurrency

- Can FILL be used to give an intuitionistic proof theory to multiparty sessions types?
  - Has multiple conclusions.
  - Pairing FILL with the work of Melliès on tensorial negation and chiralities is intuitionistic duality enough?

Lorenzen Games

- Lorenzen Games go back to the 1950's due to Lorenzen, Felscher and Rahman.
    - Rahman mentions that one could adopt a particular structural rule that enforces intuitionism.
    - No known soundness and completeness proofs are known for this semantics.
- If we adopt this rule do we obtain a sound and complete semantics for FILL?

Thank You!

Agda Code: https://github.com/heades/cut-fill-agda

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