Dialectica Categories for the Lambek Calculus

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"Why are there no dialectica models or adjoint models for non-commutative linear logic?"

Amsterdam Logic Colloquium 1991

"Valeria de Paiva. A Dialectica model of the Lambek calculus. In 8th Amsterdam Logic Colloquium, 1991."

Computational Linguistics Community

“Can we extend the Lambek Calculus with a modality that does for the structural rule of (exchange) what the modality of course ‘!’ does for the rules of (weakening) and (contraction).”

Morrill et. al

Lambek Calculus

$$\frac{}{A \vdash A}\ \textsc{ax} \qquad \frac{}{\cdot \vdash I}\ \textsc{Ur} \qquad \frac{\Gamma_2 \vdash A \qquad \Gamma_1, A, \Gamma_3 \vdash B}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash B}\ \textsc{cut} \qquad \frac{\Gamma_1, \Gamma_2 \vdash A}{\Gamma_1, I, \Gamma_2 \vdash A}\ \textsc{Ul}$$

$$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, A \otimes B, \Gamma' \vdash C}\ \textsc{Tl} \qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \textsc{Tr}$$

$$\frac{\Gamma_2 \vdash A \qquad \Gamma_1, B, \Gamma_3 \vdash C}{\Gamma_1, A \rightharpoonup B, \Gamma_2, \Gamma_3 \vdash C}\ \textsc{IRl} \qquad \frac{\Gamma_2 \vdash A \qquad \Gamma_1, B, \Gamma_3 \vdash C}{\Gamma_1, \Gamma_2, B \leftharpoondown A, \Gamma_3 \vdash C}\ \textsc{ILl}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightharpoonup B}\ \textsc{IRr} \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash B \leftharpoondown A}\ \textsc{ILr}$$

Lambek Calculus

$$\frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, A \otimes B, \Gamma' \vdash C}\ \textsc{Tl} \qquad\qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\ \textsc{Tr}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B} \qquad \dfrac{}{A \vdash A}}{B, A \vdash B \otimes A}}{A, B \vdash B \otimes A}}{A \otimes B \vdash B \otimes A}$$

Lambek Calculus

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash B \leftharpoonup A}\ \mathrm{IL}_{\mathrm{R}} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightharpoonup B}\ \mathrm{IR}_{\mathrm{R}}$$

Lambek Calculus

"Elise" has type n

"Elise works" has type s

"works" has type $s \leftharpoonup n$

"(Elise works) here" has type s

"here" has type $s \leftharpoonup s$

"Elise (never works)" has type s

"never" has type $(s \leftharpoonup n) \rightharpoonup (s \leftharpoonup n)$

Lambek Calculus

"and" has type $s \rightharpoonup (s \leftharpoonup s)$

But, isn't "and" commutative in English?

Lambek Calculus with Exchange

$$\frac{\Gamma_1, A, \Gamma_2 \vdash B}{\Gamma_1, \kappa A, \Gamma_2 \vdash B}\ \text{EL} \qquad \frac{\kappa\Gamma \vdash B}{\kappa\Gamma \vdash \kappa B}\ \text{ER}$$

Lambek Calculus with Exchange

$$\frac{\Gamma_1, A, \kappa B, \Gamma_2 \vdash C}{\Gamma_1, \kappa B, A, \Gamma_2 \vdash C} \text{ E2}$$

$$\frac{\Gamma_1, \kappa A, B, \Gamma_2 \vdash C}{\Gamma_1, B, \kappa A, \Gamma_2 \vdash C} \text{ E1}$$

Lambek Calculus with Exchange

"and" has type $s \rightharpoonup (s \leftharpoonup s)$

Lambek Calculus with Exchange

$$\text{“and” has type } s \rightharpoonup (s \leftharpoonup \kappa s)$$

$$s \rightharpoonup (s \leftharpoonup \kappa s) \Leftrightarrow (\kappa s \otimes s) \rightharpoonup s$$

Original Dialectica Construction

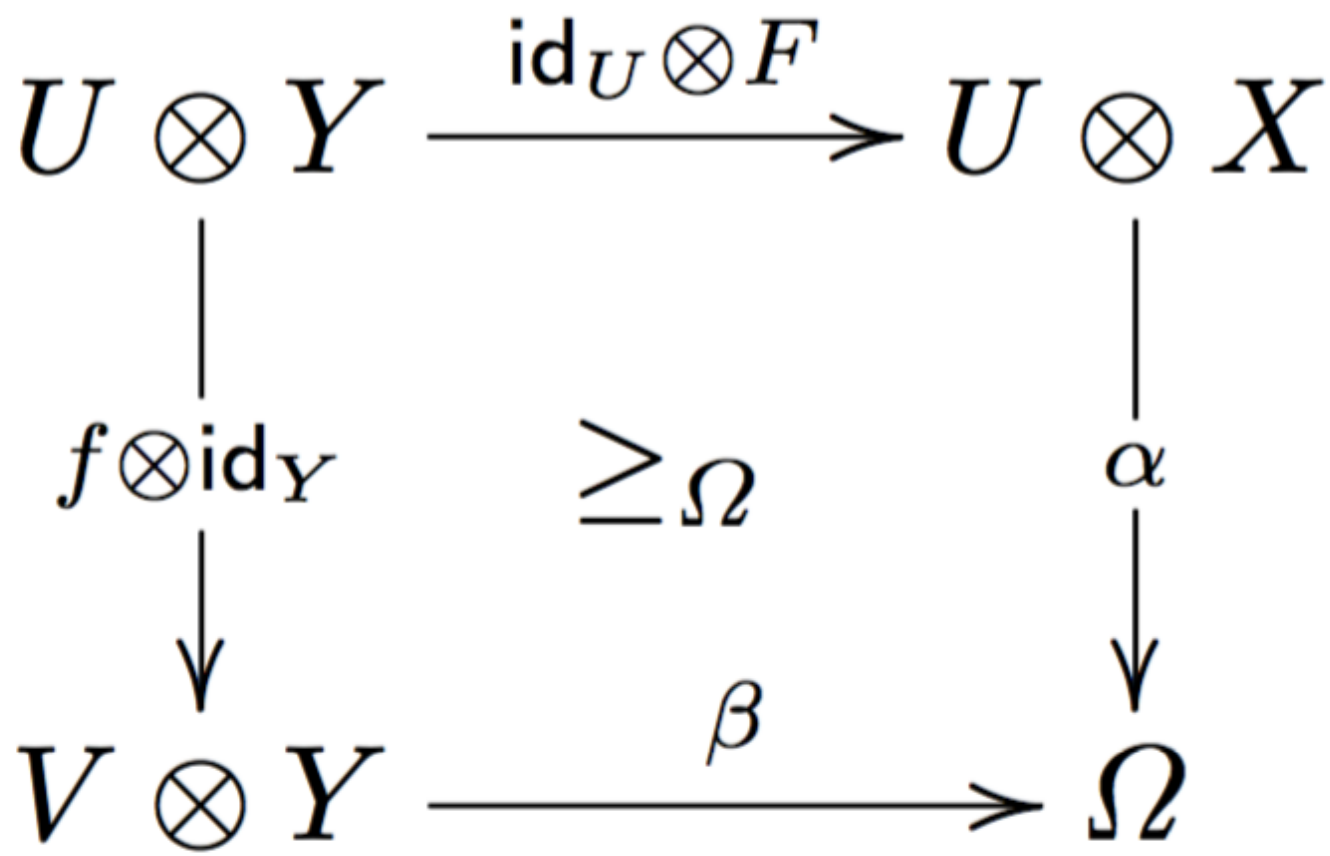
Suppose $\mathcal{C}$ is a symmetric monoidal closed category, and $\Omega \in \mathsf{Obj}(C)$ is a lineale $(\Omega, \multimap, \cdot, \leq, e)$. Then the category $\mathsf{Dial}_\Omega(C)$ is defined as follows:

Objects: (U, X, α) where $U, X \in \mathsf{Obj}(C)$ and $\alpha : U \otimes X \longrightarrow \Omega$

Morphisms: $(f, F) : (U, X, \alpha) \longrightarrow (V, Y, \beta)$ where $f \in \mathsf{Hom}_C(U, V)$ and $F \in \mathsf{Hom}_C(Y, X)$ such that:

Dialectica Categories

$$\forall u \in U. \forall y \in Y. \alpha(u, F(y)) \leq_{\Omega} \beta(f(u), y)$$



Dialectica Categories

- Full Intuitionistic Linear Logic:
  - Multiplicatives: Tensor and Par
  - Additives: Products and Coproducts
  - Modalities: of-course (!) and why-not (?)

Lambek Dialectica Spaces

Suppose $(M, \leq, \circ, e, \rightharpoonup, \leftharpoonup)$ is a biclosed poset. Then we define the category of **dialectica Lambek spaces**, $\mathsf{Dial}_M(\mathsf{Set})$, as follows:

Objects: (U, X, α) where $U, X \in \mathsf{Obj}(\mathsf{Set})$ and $\alpha : U \times X \longrightarrow M$

Morphisms: $(f, F) : (U, X, \alpha) \longrightarrow (V, Y, \beta)$ where $f \in \mathsf{Hom}_{\mathsf{Set}}(U, V)$, and $F \in \mathsf{Hom}_{\mathsf{Set}}(Y, X)$ s.t.

$$\forall u \in U. \forall y \in Y. \alpha(u, F(y)) \leq \beta(f(u), y)$$

Lambek Dialectica Spaces: Tensor Product

$$(U, X, \alpha) \otimes (V, Y, \beta) = (U \times V, (V \to X) \times (U \to Y), \alpha \otimes \beta)$$

$$(\alpha \otimes \beta)((u, v), (f, g)) = \alpha(u, f(v)) \circ \beta(g(u), v)$$

Lambek Dialectica Spaces: Internal Homs

$$(V, Y, \beta) \leftharpoonup (U, X, \alpha) = ((U \to V) \times (Y \to X), U \times Y, \alpha \leftharpoonup \beta)$$

$$(U, X, \alpha) \rightharpoonup (V, Y, \beta) = ((U \to V) \times (Y \to X), U \times Y, \alpha \rightharpoonup \beta)$$

$$\mathsf{Hom}(A \otimes B, C) \cong \mathsf{Hom}(A, B \rightharpoonup C)$$

$$\mathsf{Hom}(A \otimes B, C) \cong \mathsf{Hom}(B, C \leftharpoonup A)$$

Lambek Dialectica Spaces: of-course Modality

$$!(U, X, \alpha) = (U, U \to X^*, !\alpha)$$

$$(!\alpha)(u, f) = \alpha(u, x_1) \circ \cdots \circ \alpha(u, x_i)$$

where $f(u) = (x_1, \ldots, x_i)$

Lambek Dialectica Spaces: of-course Modality

$$\varepsilon_! : !A \longrightarrow A$$

$$\delta_! : !A \longrightarrow !!A$$

$$e : !A \longrightarrow I$$

$$d : !A \longrightarrow !A \otimes !A$$

Lambek Dialectica Spaces: exchange Modality

$$\kappa(U, X, \alpha) = (U, X, \kappa\alpha)$$

$$(\kappa\alpha)(u, x) = \kappa(\alpha(u, x))$$

Lambek Dialectica Spaces: exchange Modality

$$\varepsilon_{\kappa} : \kappa A \longrightarrow A$$

$$\delta_{\kappa} : \kappa A \longrightarrow \kappa\kappa A$$

$$\beta L : \kappa A \otimes B \longrightarrow B \otimes \kappa A$$

$$\beta R : A \otimes \kappa B \longrightarrow \kappa B \otimes A$$

Three Lambek Calculi

- Lambek Calculus
- Lambek Calculus + of-course modality
- Lambek Calculus + exchange modality
- Lambek Calculus + both

Three Lambek Calculi

Type Theories for each:

- strongly normalizing
- confluent

Agda Dialectica Space Library

https://github.com/heades/dialectica-spaces/tree/Lambek

Thank you!